Introduction to the Almost Periodic Functions of Bohr

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0. *Introduction*

The content of this paper was presented at the Centenary of Harald Bohr with the purpose of serving as an introduction for the many non-specialists present. It is our hope that this written version will incourage the reader to study the work of Harald Bohr. The collected mathematical works appeared in 1952, cf. [8], and at the occasion of the Centenary his mathematical papers with a pedagogical aim – written in Danish – have been published, cf. [9].

In the following we will concentrate on Bohr's main results about almost periodic functions, but we shall briefly indicate how he was led to the theory and how it later merged into the theory of harmonic analysis on locally compact abelian groups. The so-called Bohr compactification of a group has become a standard concept in harmonic analysis.

The readers interested in a further study of almost periodic functions are referred to the many monographs on the subject, e.g. Amerio and Prouse [1], Besicovič [3], Bohr [7], Corduneanu [10], Maak [11]. A complete bibliography on almost periodic functions from 1923 to march 1987 has been collected, see [13].

1. *Background*

Harald and the two years older brother Niels were sons of the professor of physiology Christian Bohr, and from their youth they felt veneration for science and were acquainted with the scientists of the time. Harald began to study mathematics at the University of Copenhagen at the age of 17, and already in 1910 he defended his doctoral dissertation ([5]) on the summability theory of Dirichlet series, that is series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z} = \sum_{n=1}^{\infty} a_n e^{-(\log n)z}, \tag{1}$$

where (a_n) is a sequence of complex coefficients, and $z = x+iy$ is a complex variable. Jensen had shown i 1884 that there is an abscissa of convergence γ_0 such that (1) is convergent for $x > \gamma_0$, divergent for $x < \gamma_0$.

Bohr showed that there is a decreasing sequence $\gamma_0 \geq \gamma_1 \geq \gamma_2 \geq$ of abscissas of summability such that (1) is Cesàro summable of order r for $x > \gamma_r$ but not for $x < \gamma_r$.

Furthermore, the width $w_r = \gamma_{r-1} - \gamma_r$ of the strip $\gamma_r < x < \gamma_{r-1}$, where the series is summable of order r but not of order $r-1$, satisfies

$$1 \geq w_1 \geq w_2 \geq \dots \quad . \tag{2}$$

Bohr could furthermore show that the inequalities (2) were characteristic for the sequence of summability abscissas because, for given numbers $\gamma_0 \geq \gamma_1 \geq \dots$ such that (2) holds, he constructed a Dirichlet series having these numbers as abscissas of summability. The sum of the series (1) is a holomorphic function f in the halfplane $x > \gamma_0$. By the Cesàro summability f has a holomorphic continuation to the half-plane $x > \lim_{r\to\infty} \gamma_r$. Bohr also showed the remarkable result that $\lim_{r\to\infty} \gamma_r$ is characterized as the infimum of the numbers α for which f has a holomorphic extension to the half-plane $x > \alpha$ satisfying an estimate

$$|f(x+iy)| \leq A + |y|^B,$$

where A,B depend on α.

About the same time the Hungarian mathematician Marcel Riesz had examined the summability theory of general Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \tag{3}$$

where (λ_n) is a sequence of real numbers. Bohr had also considered this general case, but in the dissertation he restricted the investigations to the special case of $\lambda_n = -\log n$.

As a result of his investigations on dirichlet series Bohr got into fruitful collaboration with Landau in Göttingen about the Riemann zeta function.

For a period of several years partially overlapping with the first world war Bohr was engaged in writing a treatise in Danish on mathematical analysis together with professor Mollerup. Bohr knew the famous Cours d'Analyse of Jordan from his years of study and he was very much influenced by it. The mathematical analysis textbook of Bohr and Mollerup should get an enormous influence on the teaching of mathematics in Denmark, and it was used from 1915 to the 1960'ies both at the University of Copenhagen and at the Technical University, although in revised editions. Further information about the life and work of Bohr can be found in his own lecture "Looking backwards" and in the memorial address by B. Jessen, both published in the collected mathematical works [8].

2. *Almost periodic functions*

It was after the completion of the mathematical analysis textbook that Bohr took up the investigations which should eventually lead to his main accomplishment, the theory of almost periodic functions. The starting point was an attempt to characterize the functions $f(z)$ which admit a representation by a Dirichlet series (3).

On a vertical line $z = x_0+iy$ this leads to the representation of a function $f(x_0+iy)$ of a real variable y as sum of a series

$$\sum_{n=1}^{\infty} b_n e^{i\lambda_n y} \text{ where } b_n = a_n e^{\lambda_n x_0}.$$

Such series comprise Fourier series for periodic functions with period $p > 0$ corresponding to $\lambda_n = \frac{2\pi}{p} n$, $n \in \mathbf{Z}$. Bohr's main contribution was to give an intrinsic characterization of the class of functions $f: \mathbf{R} \to \mathbf{C}$ which can be uniformly approximated by *trigonometric polynomials,*

$$\sum_{n=1}^{N} a_n e^{i\lambda_n y}, \tag{4}$$

where the frequences λ_n can be arbitrary real numbers, and the coefficients a_n arbitrary complex numbers.

He proved that the uniform closure of the trigonomtric polynomials are those continuous functions which are *almost periodic* in a sense explained below.

If $f: \mathbf{R} \to \mathbf{C}$ is a function of a real variable and $\varepsilon > 0$, then $\tau \in \mathbf{R}$ is called a *translation number* or an *almost period* for f corresponding to ε if

$$|f(x+\tau)-f(x)| \leq \varepsilon \quad \text{for all} \quad x \in \mathbf{R}.$$

A subset $A \subseteq \mathbf{R}$ is called *relatively dense* in $\mathbf{R}$, if there exists a sufficiently big number $l > 0$ such that every interval of length l contains at least one number from A.

Finally a continuous function $f: \mathbf{R} \to \mathbf{C}$ is called *almost periodic,* if for every $\varepsilon > 0$ the set $\{\tau_f(\varepsilon)\}$ of translation numbers for f corresponding to ε is relatively dense.

In other words, a continuous function *f is almost periodic if to every $\varepsilon > 0$ there corresponds a number $l = l(\varepsilon) > 0$ such that any interval of length l contains at least one number τ such that*

$$|f(x+\tau)-f(x)| \leq \varepsilon \quad \textit{for all} \quad x \in \mathbf{R}.$$

A continuous periodic function is almost periodic since a period p is a translation number corresponding to any $\varepsilon > 0$. If f is an almost periodic function which is non-periodic, and if $l(\varepsilon)$ denotes the smallest possible length corresponding to $\varepsilon > 0$, then $l(\varepsilon)$ will increase to infinity as ε decreases to zero. In fact if $l(\varepsilon) \leq l$ for all $\varepsilon > 0$, then the interval $[l,2l]$ contains a sequence (τ_n) such that τ_n is a translation number corresponding to $\frac{1}{n}$. Any accumulation point for the sequence (τ_n) is a period for f.

The first basic result in the theory is easy to prove: An almost periodic function is *uniformly continuous and bounded.*

The set $\mathcal{AP}$ of almost periodic functions is stable under addition and multiplication, so $\mathcal{AP}$ is an algebra of functions. More generally if $f_1,\dots,f_n : \mathbf{R} \to \mathbf{C}$ are almost periodic and $\varphi : \mathbf{A} \to \mathbf{C}$ is a continuous function *defined on a subset* $\mathbf{A} \subseteq \mathbf{C}^n$ such that

$$\text{closure } \{(f_1(x),\dots,f_n(x)) \mid x \in \mathbf{R}\} \subseteq \mathbf{A},$$

then $\varphi(f_1(x),\dots,f_n(x))$ is again almost periodic.

This is not so obvious and uses the fact that there exists for every $\varepsilon > 0$ a relatively dense set of *common* translation numbers for $f_1,\dots,f_n$ corresponding to ε.

The principal concept for the further development of the theory is the *mean value* of an almost periodic function f. Bohr proved that the number

$$\frac{1}{T}\int_a^{a+T} f(x)\,dx$$

has a limit as T tends to infinity, even uniformly for $a \in \mathbf{R}$. This limits is called the mean value of f and is denoted $\mathcal{M}\{f\}$.

It is easy to see that $\mathcal{M}$ is a positive linear functional on $\mathcal{AP}$, and if $f \geq 0, f \neq 0$ then $\mathcal{M}\{f\} > 0$. If we put

$$(f,g) = \mathcal{M}\{f\bar{g}\} \quad \text{for} \quad f,g \in \mathcal{AP},$$

then $(\cdot,\cdot)$ is a scalar product, turning $\mathcal{AP}$ into a pre Hilbert space with the norm $\|f\| = \sqrt{(f,f)}$. The exponentials e_λ, $\lambda \in \mathbf{R}$ defined by $e_\lambda(x) = e^{i\lambda x}$ form an orthonormal family so $\mathcal{AP}$ is a non-separable pre Hilbert space. It is not complete.

With $f \in \mathcal{AP}$ Bohr associated the orthogonal expansion

$$f \sim \sum_{\lambda \in \mathbf{R}} a_\lambda e^{i\lambda x}, \tag{5}$$

where $a_\lambda = (f,e_\lambda) = \mathcal{M}\{f(x)e^{-i\lambda x}\}$.

Sometimes $\lambda \to a_\lambda$ is called the *Bohr transform* of f. For any finite set Λ of real numbers Bessel's approximation theorem yields

$$\|f\|^2 = \|f - \sum_{\lambda \in \Lambda} a_\lambda e_\lambda\|^2 + \sum_{\lambda \in \Lambda} |a_\lambda|^2, \tag{6}$$

showing that only countably many of the numbers a_λ, $\lambda \in \mathbf{R}$ are different from zero. Therefore, the orthogonal expansion (5) has only countably many non-zero terms; it is called the *(almost periodic) Fourier series of f*. The set $S = \{\lambda \in \mathbf{R} | a_\lambda \neq 0\}$ is called the *spectrum of f*, and the numbers $\lambda \in S$ are called the *frequences* of *f*.

It is furthermore easy to see that the Fourier series of a periodic function *f* coincides with the almost periodic Fourier series of *f*.

The theory developed so far is quite elementary. The importance of the theory was underlined by the following fundamental results, the proofs of which given by Bohr were long and difficult.

The theorems are:

(A) *The uniqueness theorem.*
If $f, g \in \mathcal{AP}$ have the same Fourier series then $f = g$. Equivalently $(e_\lambda)_{\lambda \in \mathbf{R}}$ is a maximal orthonormal system in $\mathcal{AP}$.

(B) *Parseval's formula.*

$$\|f\|^2 = \sum_{\lambda \in \mathbf{R}} |a_\lambda|^2 \quad \text{for any} \quad f \in \mathcal{AP}.$$

(C) *The approximation theorem.*
For $f \in \mathcal{AP}$ and $\varepsilon > 0$ there exists a trigonometric polynomial p of the form (4) such that $|f(x) - p(x)| \leq \varepsilon$ for all $x \in \mathbf{R}$.

The theory outlined so far appeared in two long papers in Acta Mathematica from 1924 and 1925, see [6],I,II, comprising more than 200 pages. The results had been announced in two notes in Comptes Rendus de l'Academie des Sciences, Paris 1923, see [8].

The first Acta paper contains the proof of Theorem B, and Theorem A is an easy consequence of Theorem B. In the proof of Theorem B Bohr considered for $T > 0$ the piecewise continuous function f_T which is equal to f on $[0, T[$ and periodic with period T. By Parseval's formula for periodic functions one has

$$\frac{1}{T}\int_0^T |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |a_{n,T}|^2,$$

where

$$a_{n,T} = \frac{1}{T}\int_0^T f(x) e^{-in\frac{2\pi}{T}x}\, dx.$$

Via a very delicate analysis Bohr obtained the result by letting $T \to \infty$. In the second Acta paper Bohr proved the approximation theorem using periodic functions of infinitely many variables.

In 1927 Bochner gave the following very important characterization of almost periodic functions, cf. [4]:

A function $f\colon \mathbf{R} \to \mathbf{C}$ *is almost periodic if and only if it is continuous and the set of translates* $\{f(x+a) \mid a \in \mathbf{R}\}$ *has compact closure in the uniform metric.*

The importance of this result lies in the fact that the compactness characterization can be used as starting point for the more general theory of almost periodic functions on groups as developed by von Neumann in 1934. From Bochner's result it is also obvious that the sum and product of almost periodic functions are again almost periodic.

Alternative proofs of the three fundamental theorems A, B, C were given shortly after Bohr's work by many different mathematicians e.g. Bochner, de la Vallée Poussin, Weyl and Wiener. This demonstrates the enormous interest the theory raised.

In a third major paper in Acta Mathematica from 1926 ([6],III) Bohr studied analytic almost periodic functions and their corresponding Dirichlet series.

For the definition of this concept it is useful to introduce the notion of an *equi-almost periodic* family $\mathcal{F}$ of continuous functions $f\colon \mathbf{R} \to \mathbf{C}$, thereby meaning that the set of common translation numbers for the functions in $\mathcal{F}$ corresponding to $\varepsilon > 0$ is relatively dense, i.e.

$$\bigcap_{f \in \mathcal{F}} \{\tau_f(\varepsilon)\} \quad \text{is relatively dense for any } \varepsilon > 0.$$

An analytic function f in a vertical strip $\alpha < x < \beta$ in the complex plane is called almost periodic in the strip if the family $\mathcal{F} = \{f(x+iy) \mid x \in \,]\alpha,\beta[\}$ is equi-almost periodic as functions of $y \in \mathbf{R}$. It turns out that the functions in $\mathcal{F}$ have the same frequences (λ_n) and that the Fourier coefficients

$$a_n(x) = \mathcal{M}_y \{f(x+iy)e^{-i\lambda_n y}\}$$

have the form $a_n e^{\lambda_n x}$ for a constant $a_n \neq 0$, showing that the Fourier expansion has the form

$$f(x+iy) \sim \Sigma\, a_n e^{\lambda_n (x+iy)}$$

called the *Dirichlet expansion* of f.

We shall not go further into the analytic almost periodic functions, which in a sense was the goal of Bohr's investigations.

3. *The Bohr compactification*

Let us consider the theory from another point of view.

The continuous group characters of the real line, i.e. the continuous homomorphisms of $(\mathbf{R},+)$ into $(\mathbf{T},\cdot)$, where

$$\mathbf{T} = \{z \in \mathbf{C} \mid |z| = 1\},$$

are precisely the functions $(e_\lambda)_{\lambda \in \mathbf{R}}$. The coarsest topology on $\mathbf{R}$ for which these functions e_λ, $\lambda \in \mathbf{R}$ are continuous, is strictly coarser than the ordinary topology. We propose to call it the *Bohr topology*. With the Bohr topology the real line is organized as a topological group, and a basis for the neighbourhoods of zero is given by the following sets

$$[\lambda_1,\ldots,\lambda_m;\delta] = \{\tau \in \mathbf{R} \mid |e^{i\lambda_1\tau}-1| < \delta,\ldots,|e^{i\lambda_m\tau}-1| < \delta\},$$

where $m \in \mathbf{N}$, $\lambda_1,\ldots,\lambda_m \in \mathbf{R}$ and $\delta > 0$ are arbitrary.

The real line with the Bohr topology is not compact, not even locally compact, but it can be compactified. Let $\mathbf{T}_\lambda$ be a copy of the circle group for each $\lambda \in \mathbf{R}$ and let

$$j : \mathbf{R} \to \prod_{\lambda\in\mathbf{R}} \mathbf{T}_\lambda$$

be defined by

$$j(x) = (e_\lambda(x))_{\lambda\in\mathbf{R}} = (e^{i\lambda x})_{\lambda\in\mathbf{R}}.$$

The product set is a compact group under the product topology. The mapping j is clearly a homeomorphism of $\mathbf{R}$ with the Bohr topology onto the image $j(\mathbf{R})$. The closure of $j(\mathbf{R})$ is a compactification of $\mathbf{R}$ with the Bohr topology, called the *Bohr compactification* of $\mathbf{R}$ and denoted $\beta(\mathbf{R})$, i.e.

$$\beta(\mathbf{R}) = \overline{j(\mathbf{R})},$$

which is a compact group. In the sequel we identify $\mathbf{R}$ and $j(\mathbf{R})$.

By the approximation theorem an almost periodic function $f\colon \mathbf{R} \to \mathbf{C}$ is uniformly continuous in the Bohr topology, and therefore it has a unique continuous extension F to the Bohr compactification. Conversely, if $F\colon \beta(\mathbf{R}) \to \mathbf{C}$ is a continuous function on the compact group $\beta(\mathbf{R})$, then it is uniformly continuous, and so is the restriction f of F to the real line with the Bohr topology. This means that for any $\varepsilon > 0$ there exists a neighbourhood of zero of the form $[\lambda_1,\ldots,\lambda_m;\delta]$ such that

$$|f(x+\tau)-f(x)| \le \varepsilon \quad \text{for all} \quad \tau \in [\lambda_1\ldots,\lambda_m;\delta],$$

but this set is an ordinary neighbourhood of zero and relatively dense as is easily seen, so f is actually almost periodic.

This shows that there is a one-to-one correspondence between the almost periodic functions of Bohr and the continuous functions on the Bohr compactification $\beta(\mathbf{R})$.

The Bohr compactification $\beta(\mathbf{R})$ can be described as the set of all characters of $\mathbf{R}$, i.e. the set of all homomorphisms $\varphi\colon \mathbf{R} \to \mathbf{T}$. In fact, since $\beta(\mathbf{R})$ is the closure of the set of continuous characters, $\beta(\mathbf{R})$ consists of characters, and the fact that all characters belong to $\beta(\mathbf{R})$ is an easy consequence of Kroneckers's theorem.

4. *Harmonic analysis on locally compact abelian groups*

Bohr's theory of almost periodic functions has many resemblances with those of Fourier series and Fourier integrals. During the 1930'ies these three theories merged into a common theory called harmonic analysis on locally compact abelian groups. Many mathematicians contributed to this achievement e.g. Bochner, van Kampen, Pontryagin, Weil. The starting point was the theorem of Haar about the existence of an invariant measure on a locally compact group, now called Haar measure. With the publication in 1940 of Weil's fundamental monograph [12] the theory became widely known although many simplifications and refinements have appeared since then.

To every locally compact abelian group G is associated a *dual group* $\hat{G}$. As a set $\hat{G}$ concists of the continuous characters of G, i.e. the continuous homomorphisms $\gamma\colon G \to \mathbf{T}$. With pointwise multiplication and the topology of uniform convergence on compact subsets of G it turns out that $\hat{G}$ is a locally compact abelian group. It is customary to write (x,γ) in place of $\gamma(x)$ for $x \in G$, $\gamma \in \hat{G}$.

For a continuous function $f\colon G \to \mathbf{C}$ with compact support the Fourier transform $\hat{f}\colon \hat{G} \to \mathbf{C}$ is defined by

$$\hat{f}(\gamma) = \int f(x)\overline{(x,\gamma)}\, dm_G(x) \text{ for } \gamma \in \hat{G},$$

and it is possible to choose the Haar measures m_G and $m_{\hat{G}}$ on G and $\hat{G}$ in such a way that

$$\int_G |f(x)|^2 dm_G(x) = \int_{\hat{G}} |\hat{f}(\gamma)|^2 dm_{\hat{G}}(\gamma) \tag{7}$$

for all such f. This formula shows that the Fourier transformation $f \to \hat{f}$ has a unique extension to an isometry of $L^2(G)$ onto $L^2(\hat{G})$.

For $G = \mathbf{T}$ we have $\hat{G} \approx \mathbf{Z}$ and $\hat{f}(n)$ is the n'th Fourier coefficient, while (7) is Parseval's formula.

For $G = \mathbf{R}$ we have $\hat{G} \approx \mathbf{R}$, $\hat{f}$ is the ordinary Fourier transform and (7) is Plancherel's theorem.

For $G = \beta(\mathbf{R})$ and an almost periodic function $f\colon \mathbf{R} \to \mathbf{C}$ with unique continuous extension $F\colon \beta(\mathbf{R}) \to \mathbf{C}$ to the Bohr compactification $\beta(\mathbf{R})$, it turns out that

$$\mathscr{M}\{f\} = \int F dm_{\beta(\mathbf{R})} ,$$

i.e. the mean value of f is the Haar integral of the extension F. The dual group of $\beta(\mathbf{R})$ can be identified with $\mathbf{R}$ with the discrete topology, and $\hat{F}(\lambda) = a_\lambda$, the λ'th Fourier coefficient, while (7) is Parseval's formula, cf. (B) in §2.

Pontryagin's duality theorem states that the dual group of $\hat{G}$ can be identified with G, i.e. $\hat{\hat{G}} \approx G$.

Furthermore, for any locally compact abelian group G there is a Bohr compactification $\beta(G)$, which can be realized as the compact dual group of $\hat{G}$ considered as a discrete group. Again there is a one-to-one correspondence between continuous almost periodic functions on G and continuous functions on $\beta(G)$. The term Bohr compactification seems to have been introduced by Anzai and Kakutani in two papers from 1943, cf. [2].

5. *Conclusion*

We shall not attempt to describe the many generalizations and applications of the theory of almost periodic functions. The literature is enormous, cf. [13], and it would be an overwhelming task.

The other papers in this volume will shed some light on the various aspects of the subject and thereby show the richness and beauty of the theory initiated by Harald Bohr.

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The entries are listed by year from 1923-1972 with *Mathematical Reviews* or *Zentralblatt* or *Jahrbuch* numbers. The years 1973 – March 1987 are by title only with a *Mathematical Reviews* number or a *Current Mathematical Publication* number. These include those items for which the words "almost periodic" or "quasi-periodic" appear in the title.

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